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On Spectral Concentration

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Abstract.

Let $H(\varepsilon)$ be a family of selfadjoint operators and let $\triangle(\varepsilon)$ be a family of sets. We say that the spectrum of the family of operators $H(\varepsilon)$ is concentrated on the family of sets $\triangle(\varepsilon)$, if $E_{\triangle(\varepsilon)}(\varepsilon)$, the spectral projector associated with $H(\varepsilon)$ and $\triangle(\varepsilon)$, tends to the identity. In this report a set $\triangle(\varepsilon)$ is defined $[(3.5)_v]$ with the aid of a formal perturbation process, and conditions are given under which the spectrum of $H+\varepsilon V$ is concentrated on this set $\triangle(\varepsilon)$.

§ 1. Introduction

After F. Rellich developped a mathematically rigorous perturbation theory it was pointed out [3,8,9] that in general the result of perturbation procedure has an asymptotic meaning only. Nevertheless, there are many questions which one may ask about the "asymptotic spectral behavior" of a family of operators. A possible question is the following : Suppose that the spectrum of an operator suffers drastic changes during a perturbation. In what sense are the disturbed and undisturbed spactra close to each other ? In order to define this "closeness" let us start from the spectral theorem for self-adjoint operators. According to this theorem to every Borel set on the real line one may assign a spectral projector and the spectral projector assigned to the spectrum of the operator is the identity. Therefore it is reasonable to say that the spectrum of an operator is "close" to a given set if the spectral projector associated with the given set and operator is "close", say in the strong topology to the identity operator. More precisely: Let $H(\varepsilon)$ be a family of operators and let $E_{\triangle}(\varepsilon)$ denote the spectral projector of $H(\varepsilon)$ associated with the set $\triangle$. If $\triangle(\varepsilon)$ is a family of sets such that $E_{\triangle(\varepsilon)}(\varepsilon) \to 1$, then we shall say that the spectrum of the family of operators $H(\varepsilon)$ is concentrated on the family of sets $\triangle(\varepsilon)$[1]. Since this family of sets is not uniquely determined we may speak about several concentration phenomena, which are associated with the same family of operators $H(\varepsilon)$ but with different family of sets $\triangle(\varepsilon)$. For example the following "concentra-

1. This definition is an abstract version of the one suggested by E.C. Titchmarsh in [8] for differential operators.

tion statement" is an immediate corollary of a theorem of Rellich: [6. p. 369].

Let $H(\varepsilon) \to H(0)$, and let $\triangle$ be a set which is the union of a finite number of bounded or unbounded intervals, and let $\triangle$ contain the spectrum of $H(0)$ in such a way that the point-spectrum of $H(0)$ is contained in the interior of $\triangle$. Then the spectrum of the family of operators $H(\varepsilon)$ is concentrated on the family of sets consisting of the set $\triangle$.

Rellich has also shown [5] that the above statement is not necessarily true if the set $\triangle$ does not contain the undisturbed pointspectrum in its interior. Recently, however, E.C. Titchmarsh has shown [8], using the theory of differential operators, [7], that the spectrum of the family of operators entering the theory of the Stark effect is concentrated on a family of sets $\triangle(\varepsilon)$, which is close to the undisturbed spectrum, for small ε.

In this report we shall establish a concentration phenomenon, for a rather large class of abstract operators. More precisely, let the family of operators $H(\varepsilon)$ be of the form $H(\varepsilon) = H + \varepsilon V$. In order to define the set $\triangle(\varepsilon)$ let us carry out a formal perturbation procedure, and define the "apparent pointeigenvalues" as the first order approximation in the formal power series for the formal pointeigenvalues. Let δ be an arbitrary positive number. Then define $\triangle(\varepsilon)$ to be the union of the undisturbed continuous spectrum with the $\delta\varepsilon$ neighboorhoods of the "apparent pointeigenvalues". In Theorem 3.2 we shall show that under general conditions the spectrum of the family of operators $H(\varepsilon)$

is concentrated on the family of sets $\triangle(\varepsilon)$. We may visualize this concentration statement in figure 1, where $\triangle(\varepsilon)$, i.e. an approximate spectrum of $H(\varepsilon)$ is plotted versus ε. For $\varepsilon = 0$ the approximate spectrum is the exact spectrum. Figure 2 corresponds to Rellich's concentration statement.

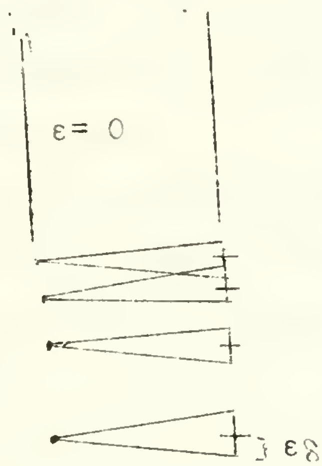

Fig. 1

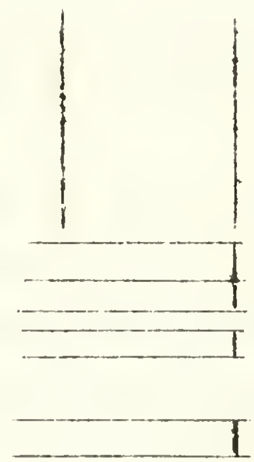

Fig. 2

In section 4 we shall show that this general concentration theorem is applicable to the family of operators entering the theory of the Stark effect of the hydrogen atom. The theorem which we obtain this way is closely related to Titchmarsh's Theorem [8] [c.f. Appendix]. Finally let us note that it is likely that this "general concentration theorem" is applicable in other cases too. This is made plausible by a result of T. Kato [4], which states that in general the Schrödinger-Hamiltonian operator is essentially selfadjoint.

§ 2. Auxiliary theorems.

Rellich's Theorem. [6. p. 369]

Let $\{A_n\}$ be a sequence of (strictly) self-adjoint operators which tend strongly to the (strictly) self-adjoint operator A

in the extended sense that A is equal to the closure of $\lim A_n$. Denoting the respective spectral families by $\{E_{n\lambda}\}$ and $\{E_\lambda\}$, we have

$$E_{n\lambda} \to E_\lambda$$

for every point λ which does not belong to the pointspectrum of A. Next we need the concept of the direct integral of a family of Hilbert spaces.

Let σ be a bounded measure defined on the real line, and let $\mathfrak{H}(\lambda)$ be a family of Hilbert spaces $-\infty < \lambda < +\infty$. For simplicity we shall say that f is a vector valued function if $f(\lambda) \in \mathfrak{H}(\lambda)$ for every λ. Then we can define an inner product space, consisting of continuous vector valued functions with compact support, with inner product

$$\langle f,g \rangle = \int \langle f(\lambda), g(\lambda) \rangle \, d\sigma(\lambda) \quad .$$

We shall call the closure of this inner product space a direct integral of Hilbert spaces[2] and we shall denote it by

$$\mathfrak{H}_{\oplus} = \int_{\oplus} \mathfrak{H}(\lambda) d\sigma(\lambda) \quad .$$

On the direct-integral space $\mathfrak{H}_{\oplus}$ we can define the multiplication operator M by the equation

$$(2.1) \qquad M\phi(\lambda) = \lambda\phi(\lambda) \qquad ,$$

with domain consisting of all functions ϕ for which $\lambda\phi(\lambda) \in \mathfrak{H}_{\oplus}$. With the aid of the direct-integral space and the multiplication operator, we can define the spectral transformation U of an

[2] The concept of the direct integralspace introduced in [2] is more general than this concept. However, it is easy to establish as a Corollary of the Remark made before Theorem 1 of [2] that the two spaces are isomorphic.

arbitrary operator A as follows.

Definition :

Let A be an arbitrary operator defined on an arbitrary Hilbert space $\mathfrak{H}$. If there is a unitary transformation U which maps $\mathfrak{H}$ into some direct-integral-space $\mathfrak{H}_{\oplus}$ in such a way that

$$A = U^{*}MU \tag{2.2}$$

holds, then U is called a spectral transformation of A.

Using this definition we can state the following:

Spectral Representation Theorem. [1,2]

Any strictly self-adjoint operator A defined on a separable Hilbert space admits a spectral transformation U. Any two spectral transformations U_1, U_2, of the same operator A are unitarily equivalent, i.e. there is a unitary transformation U mapping a direct-integral-space into another direct-integral-space such that

$$U_1 = U^{*} U_2 U \quad .$$

§ 3. The Concentration Theorem.

In this section we shall define a family of sets $\triangle(\varepsilon)$ and give conditions under which the spectrum of the family of operators $H + \varepsilon V$ is concentrated on the sets $\triangle(\varepsilon)$. First we shall establish this for the special case in which V is the multiplication operator; then we shall use the Spectral Representation Theorem to establish the concentration in the general case.

We start with a technicality. The limit of a family of strongly convergent unbounded operators, is defined as follows, ([6], p. 299): Let $H(\varepsilon)$ be an arbitrary family of operators,

then $\lim_{\varepsilon \to 0} H(\varepsilon)$ is the operator whose domain consists of those vectors ϕ for which $H(\varepsilon)\phi$, defined for every $\varepsilon < \varepsilon_0$ where ε_0 may depend on ϕ, a Cauchy sequence and

$$(\lim_{\varepsilon \to 0} H(\varepsilon))\phi = \lim_{\varepsilon \to 0} (H(\varepsilon)\phi) \quad .$$

We shall also need an obvious but important proposition concerning the limit of strongly convergent operators. Let $H(\varepsilon)$ be a family of operators satisfying the following condition:

<u>Condition 3.1</u>

$H(\varepsilon)$ is a family of formally selfadjoint operators with common dense domain $\mathfrak{H}_{H(0)}$, such that

a) for small enough ε, $H(\varepsilon)$ admits a strictly selfadjoint extension, say $\widetilde{H}(\varepsilon)$.

b) $H(\varepsilon) \longrightarrow H(0)$ on $\mathfrak{H}_{H(0)}$,

c) The operator $H(0)$ is essentially selfadjoint, i.e. its closure is strictly selfadjoint.

<u>Proposition 3.2</u>

If the family of operators $H(\varepsilon)$ satisfies Condition (3.1) then the closure of $\lim_{\varepsilon \to 0} \widetilde{H}(\varepsilon)$, (where $\widetilde{H}(\varepsilon)$ is an arbitrary selfadjoint extension of $H(\varepsilon)$) equals the closure of $H(0)$. I.e. if $\widetilde{H} = \lim_{\varepsilon \to 0} H(\varepsilon)$, then

$$\overline{(\widetilde{H})} = \overline{H(0)}$$

Proof

Clearly $H(0) \subset \widetilde{H}$, hence $\overline{H(0)} \subset \overline{(\widetilde{H})}$.

On the other hand it is clear that $\widetilde{H}$ and hence $\overline{(\widetilde{H})}$ are formally selfadjoint. Since $H(0)$ was assumed to be essentially self-

adjoint, $\overline{H(0)}$ has no proper formally selfadjoint extensions, therefore $\overline{H(0)} = (\tilde{\tilde{H}})$, which establishes Proposition 3.2.

Let $\mathfrak{H}$ be an arbitrary direct-integral-space and let M be the multiplication operator over this direct-integral-space, i.e. the operator defined by

$$M\psi(\lambda) = \lambda\psi(\lambda) \quad . \tag{2.1}$$

Let K be an essentially selfadjoint operator over $\mathfrak{H}$ satisfying the following two conditions:

<u>Condition (3.3)</u>

a) The point spectrum of $\overline{K}$ is discrete in the sense that every point eigenvalue λ_i has a neighborhood, say Δ_i, which contains no other point of the spectrum of $\overline{K}$, and each point eigenvalue is of finite multiplicity.

b) The continuous spectrum of $\overline{K}$, Δ , is the union of a finite number of bounded or unbounded intervals.

<u>Condition (3.4)</u>

The point eigenfunctions of $\overline{K}$ belong to the domain of M.

Next we turn to the description of the set $\Delta(\varepsilon)$ which will have the property that the spectrum of the family of operators $K + \varepsilon M$ is concentrated on $\Delta(\varepsilon)$. The "apparent point-eigenvalues" i.e. the first order approximations in the formal power series for the formal pointeigenvalues will enter the description of $\Delta(\varepsilon)$. In case λ_i is simple, then it is easily seen that the apparent pointeigenvalue corresponding to λ_i is $\lambda_i + \varepsilon < k_i\ Mk_i >$ where k_i is the eigenfunction of K with eigenvalue λ_i. Then Condition 3.4 insures that this expression is well

defined. In case λ_i is degenerate we have to proceed with more care. In view of Condition 3.3 the multiplicity of λ_i is finite, and according to [10] the first order approximations in the formal power series, for the formal split pointeigenvalues are given by

$$\lambda_i + \varepsilon < k_{ij}\, Mk_{ij} > \quad .$$

Here the k_{ij} $j = 1, \dots n_i$, are the eigenfunctions of the finite dimensional operator $E_{\lambda_i} M E_{\lambda_j}$, and E_{λ_i} is the projector on the eigenspace of K with eigenvalue λ_i. Now we formulate:

Definition 3.5

$$\triangle(\varepsilon) = \triangle \cup \bigcup_{\substack{i=1 \\ 1\leq j\leq n_i}}^{\infty} \delta_{ij}(\varepsilon)$$

where $\triangle$ is the continuous spectrum of the undisturbed operator, and the $\delta_{ij}(\varepsilon)$ are the $\varepsilon\delta$ neighborhoods of the split apparent pointeigenvalues, i.e. $\delta_{ij}(\varepsilon)$ is the interval defined by,

$$(3.6) \quad \delta_{ij}(\varepsilon) = [\lambda_i + \varepsilon < k_{ij}\, Mk_{ij} > - \varepsilon\delta \;,\; \lambda_i + \varepsilon < k_{ij}\, Mk_{ij} > + \varepsilon\delta] \;.$$

Here δ is an arbitrary but fixed positive number.

Now we maintain that the spectrum of the family of operators $K + \varepsilon M$ is concentrated on the family of sets $\triangle(\varepsilon)$. More precisely the following theorem holds:

<u>Theorem 3.1</u>

Let $K(\varepsilon) = K + \varepsilon M$ be a family of operators satisfying conditions (3.1), (3.3) (3.4) and let $\widetilde{K}(\varepsilon)$ denote an arbitrary self-adjoint extension of $K(\varepsilon)$. Then the spectrum of the family of

operators $\tilde{K}(\varepsilon)$ is concentrated on the family of sets $\Delta(\varepsilon)$ defined in 3.5. In other words, let $E_{\Delta(\varepsilon)}(\varepsilon)$ denote the spectral projector of $\tilde{K}(\varepsilon)$ associated with the set $\Delta(\varepsilon)$. Then

$$E_{\Delta(\varepsilon)}(\varepsilon) \to 1 \quad , \quad \text{as } \varepsilon \to 0$$

where 1 denotes the identity operator of $\tilde{\mathfrak{H}}$.

Proof.

First let us notice that by virtue of Proposition 3.2 Rellich's Theorem applies to the above family of strictly selfadjoint operators; hence

$$E_{\Delta}(\varepsilon) \longrightarrow E_{\Delta}(0) \tag{3.7}$$

or $E_{\Delta}(\varepsilon) \longrightarrow 1$, on $\mathcal{E}_{\Delta}(0)$, the eigenspace of $\bar{K}$ associated with the set Δ. Clearly $1 \geq E_{\Delta(\varepsilon)}(\varepsilon) \geq E_{\Delta}(\varepsilon)$; hence (3.7) yields

$$E_{\Delta(\varepsilon)}(\varepsilon) \longrightarrow 1, \quad \text{on} \quad \mathcal{E}_{\Delta}(0)$$

as $\varepsilon \longrightarrow 0$.

Now we turn to the proof of the statement that $E_{\Delta(\varepsilon)}(\varepsilon) \xrightarrow{s} 1$, on $\mathcal{E}_{\Delta}(0)^{\perp}$, the orthocomplement of $\mathcal{E}_{\Delta}(0)$. By virtue of the (strictly) selfadjoint character of $\bar{K}$ and condition 3.3 we have $\mathcal{E}_{\Delta}(0)^{\perp} = \mathcal{E}_{pt}(0)$, the latter being the point eigenspace of $\bar{K}$. We maintain that in order to establish

$$E_{\Delta(\varepsilon)}(\varepsilon) \longrightarrow 1 \text{ on } \mathcal{E}_{pt}(0) \quad \text{as } \varepsilon \longrightarrow 0 \tag{3.8}$$

it suffices to establish that

$$(3.8)_i \qquad E_{\Delta(\varepsilon)}(\varepsilon)k_{ij} \longrightarrow k_{ij} \quad \text{as } \varepsilon \longrightarrow 0 \qquad 1 \le j \le n_i \quad i = 1,2,3 \ldots$$

This is clear since the set of finite linear combinations of the functions k_{ij} is dense in $\mathcal{E}_{pt}(0)$, and a uniformly bounded sequence of operators converges strongly on the entire space if it converges strongly on a dense set. For brevity let us drop the second index in k_{ij} and from now on let k_i denote any of the functions k_{ij} $1 \le j \le n_i$.

We prove the remaining part of the theorem i.e. relation (3.8) by proving $(3.8)_i$. According to the Spectral Representation Theorem there is a direct integral space $\mathfrak{H}^{(\varepsilon)}_{\oplus}$, and a unitary transformation U_ε which maps $\mathfrak{H}$ into $\mathfrak{H}^{(\varepsilon)}_{\oplus}$, and for which

$(2.1)_\varepsilon$ $\quad U_\varepsilon(K+\varepsilon M) = M_\varepsilon U_\varepsilon$, where M_ε denotes the multiplication operator over $\mathfrak{H}^{(\varepsilon)}_{\oplus}$.

Hence

$$U_\varepsilon(K+\varepsilon M)k_i = M_\varepsilon U_\varepsilon k_i$$

and

$$(3.9) \qquad \varepsilon U_\varepsilon M k_i = (M_\varepsilon - \lambda_i)U_\varepsilon k_i \quad .$$

Now

$$(3.10) \qquad Mk_i = \langle k_i Mk_i \rangle \, k_i + Mk_i^{\perp} \quad ,$$

where $Mk_i^{\perp}$ is the projection of Mk_i on the orthocomplement of k_i. Insertion of (3.10) in (3.9) yields

$$\varepsilon U_\varepsilon (Mk_i^{\perp}) = (M_\varepsilon - \lambda_i - \varepsilon < k_i Mk_i >)U_\varepsilon k_i \quad .$$

Let $\eta_{\triangle_i}$ denote the characteristic function of the set $\triangle_i$, defined in $(3.3)_a$. Multiplying the previous equation by $\eta_{\triangle_i}$ we have

$$(3.11) \qquad \varepsilon\, \eta_{\triangle_i} U_\varepsilon(Mk_i^{\perp}) = \eta_{\triangle_i}(M_\varepsilon - \lambda_i - \varepsilon < k_i Mk_i >)\, U_\varepsilon k_i \quad .$$

Now observe that

$$\| \eta_{\triangle_i} U_\varepsilon (Mk_i^{\perp}) \| = \| U_\varepsilon^* \eta_{\triangle_i} (Mk_i^{\perp}) \|$$

$$(3.12) \qquad = \| E_{\triangle_i}(\varepsilon)(Mk_i^{\perp}) \| \quad ,$$

and that according to Rellich's Theorem

$$(3.13) \qquad E_{\triangle_i}(\varepsilon) \longrightarrow E_{\triangle_i}(0) \quad .$$

We recall that k_i was defined as an eigenvector of the operator $E_{\lambda_i} M E_{\lambda_i}$. Now in view of condition $(3.3)_0$ $E_{\triangle_i}(0) = E_{\lambda_i}$, hence k_i is an eigenfunction of the operator $E_{\triangle_i}(0) M E_{\triangle_i}(0)$. Therefore from (3.10) we have

$$E_{\triangle_i}(0)(Mk_i^{\perp}) = 0 \quad .$$

Upon insertion of this relation in (3.13) and (3.12) we obtain:

$$(3.14) \qquad \| \eta_{\triangle_i} U_\varepsilon (Mk_i^{\perp}) \| \longrightarrow 0$$

$$\text{as } \varepsilon \longrightarrow 0$$

On the other hand, by the definition of $\delta_i(\varepsilon)$

$$\int_{\delta_i'(\varepsilon)} ||\, U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda) \leq \int_{\delta_i'(\varepsilon)} \left| \frac{\lambda-\lambda_i-\varepsilon < k_i Mk_i >}{\varepsilon\delta} \right|^2 ||U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda)$$

where $\delta_i'(\varepsilon) = \overline{\delta_i(\varepsilon)} \cap \Delta_i$, and $\overline{\delta_i(\varepsilon)}$ is the complement of $\delta_i(\varepsilon)$.

Using (3.11) this in turn yields,

$$\int_{\delta_i'(\varepsilon)} ||U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda) \leq \delta^{-2} \int_{\Delta_i} ||U_\varepsilon(Mk_i)(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda)$$

whence (3.14) gives

$$\text{(3.15)} \qquad \int_{\delta_i'(\varepsilon)} ||U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda) = o(1) \qquad \text{at } \varepsilon = 0 \;.$$

Let $\overline{\Delta_i}$ denote the complement of Δ_i , then

$$\int_{\overline{\Delta_i}} ||U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda) = \int ||\,\eta_{\overline{\Delta_i}}(\lambda) U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda)$$

$$= ||U_\varepsilon^* \eta_{\overline{\Delta_i}} \, U_\varepsilon k_i|| = ||E_{\overline{\Delta_i}}(\varepsilon) k_i \, ||$$

and by Rellich's Theorem $||E_{\overline{\Delta_i}}(\varepsilon)k_i|| = o(1)$

hence

$$\text{(3.16)} \qquad \int_{\overline{\Delta_i}} ||U_\varepsilon k_i(\lambda)||^2 d\,\rho_\varepsilon(\lambda) = o(1) \;.$$

Relations (3.15) and (3.16) yield

$$\text{(3.17)} \qquad \int_{\overline{\delta_i(\varepsilon)}} ||U_\varepsilon k_i(\lambda)||^2 \, d\,\rho_\varepsilon(\lambda) = o(1)$$

Now clearly

$$||(1 - E_{\triangle(\varepsilon)}(\varepsilon))k_i|| = ||U_\varepsilon^* (1 - \eta_{\triangle(\varepsilon)})U_\varepsilon k_i|| =$$

(3.18)

$$= ||(1 - \eta_{\triangle(\varepsilon)})U_\varepsilon k_i|| \leq \int_{\overline{\delta_i(\varepsilon)}} ||U_\varepsilon k_i(\lambda)||^2 \, d\rho_\varepsilon(\lambda)$$

Relations (3.17) and (3.18) establish relations $(3.8)_i$, which in turn completes the proof of Theorem 3.1.

Having established Theorem 3.1 let us remove the condition that the second summand of the family of operators entering the theorem is "diagonalized". Let $H(\varepsilon) = H + \varepsilon V$ be a family of operators satisfying the following conditions:

Condition $(3.1)_V$

The operators H and V are essentially self-adjoint in some common, dense domain, say $\mathfrak{H}_H$. $H(\varepsilon) = H + \varepsilon V$ is a family of formally self-adjoint operators which, for small enough ε, admit a strictly self-adjoint extension say $\widetilde{H(\varepsilon)}$.

Condition $(3.3)_V$

a) The pointspectrum of $\overline{H}$ is discrete in the sense that every pointeigenvalue λ_i, has a neighborhood say $\triangle_i$, which contains no other point of the spectrum of $\overline{H}$, and each point eigenvalue is of finite multiplicity.

b) The continuous spectrum of $\overline{H}$, $\triangle$, is the union of a finite number of bounded or unbounded intervals.

Condition $(3.4)_V$

The point eigenfunctions of $\overline{H}$ belong to the domain of $\overline{V}$.

We define the "concentration set" $\triangle(\varepsilon)$ for the family of operators $H + \varepsilon V$ as we did it for the family $K + \varepsilon M$, i.e. by replacing K by H and M by V, in (3.5). This yields:

Definition $(3.5)_v$ –

$$\triangle(\varepsilon) = \triangle \cup \bigcup_{\substack{i=1 \\ 1\le j\le n_i}}^{\infty} \delta_{ij}(\varepsilon)$$

where $\triangle$ is the continuous spectrum of $\overline{H}$ and $\delta_{ij}(\varepsilon)$ are the $\varepsilon\delta$ neighborhoods of the apparent split pointeigenvalues, i.e. the intervals defined by

$$(3.6)_v \quad \delta_{ij}(\varepsilon) = [\lambda_i + \varepsilon < k_{ij} V k_{ij} > - \varepsilon\delta,\ \lambda_i + \varepsilon < k_{ij} V k_{ij} > \varepsilon\delta] \ .$$

Here δ is an arbitrary but fixed positive number and k_{ij} $j=1,..n_i$, are the eigenfunctions of the finite dimensional operator $E_{\lambda_i} V E_{\lambda_i}$, where E_{λ_i} is the projector on the eigenspace of H with eigen - value λ_i.

According to the Spectral Representation Theorem there is a direct integral space $\mathfrak{H}^{(v)}$, and a unitary transformation U_v such that

$$(2.2)_v \qquad \overline{V} = U_v^{*} \overline{M}_v U_v$$

where M_v denotes the multiplication operator over $\mathfrak{H}^{(v)}$. Now let us define the new family of operators over $\mathfrak{H}^{(v)}$, by

$$(3.19) \qquad U_v H U_v^{*} + \varepsilon M_v = K + \varepsilon M_v \quad .$$

Since a unitary transformation between any two Hilbert spaces

preserves the structure of the Hilbert space we have that:

If the family of operators $(H + \varepsilon V)$ satisfies condition $(3.1)_v$ with domain $\mathfrak{H}_H \subset \mathfrak{H}$ then the family of operators $K + \varepsilon M_v$ satisfies condition $(3.1)_v$ with domain $\mathfrak{H}_K = U_v \mathfrak{H}_H \subset \mathfrak{H}^v$.

Also it is clear that if the family of operators $H + \varepsilon V$ satisfies conditions $(3.3)_v$ and $(3.4)_v$ then the family $K + \varepsilon M_v$ satisfies conditions (3.3) and (3.4). Therefore Theorem 3.1 applies to the family of operators $K + \varepsilon M_v$. Now, $E^v(\varepsilon)$, the spectral family of $H + \varepsilon V$, can be obtained from $E(\varepsilon)$, the spectral family of $K + \varepsilon M_v$, with the aid of the formula

$$E^v(\varepsilon) = U_v^* \, E(\varepsilon) \, U_V \quad .$$

In view of these facts we can state:

<u>Theorem 3.2 (Concentration Theorem).</u>

Let $H + \varepsilon V$ be a family of formally self-adjoint operators satisfying conditions $(3.1)_v$ $(3.3)_v$ $(3.4)_v$ and let $\widetilde{H + \varepsilon V}$ denote an arbitrary (strictly) self-adjoint extension of $H + \varepsilon V$. Then the spectrum of the family of operators $\widetilde{H + \varepsilon V}$ is concentrated on the set $\triangle(\varepsilon)$, where $\triangle(\varepsilon)$ is defined in $(3.5)_v$. In other words let $E^{(v)}_{\triangle(\varepsilon)}(\varepsilon)$ denote the spectral family of $H + \varepsilon V$, associated with $\triangle(\varepsilon)$. Then

$$E^{(v)}_{\triangle(\varepsilon)}(\varepsilon) \longrightarrow 1 \, , \qquad \text{as} \quad \varepsilon \longrightarrow 0.$$

§ 4. Application of the Concentration Theorem to the Stark Effect.

Let the undisturbed operator be the Hamiltonian energy operator of the hydrogen atom, i.e. the operator defined by

$$H\phi = -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} - \frac{1}{r}\phi$$

(4.1)

$$= -\triangle\phi - \frac{1}{r}\phi \quad , \quad r^2 = x^2 + y^2 + z^2$$

and let the disturbing operator εV be the potential-operator of a weak homogeneous, electric field in the Z direction, i.e.

(4.2) $$\varepsilon V\phi(x,y,z) = \varepsilon z\phi(x,y,z) \quad .$$

Both operators are defined on $\mathfrak{D}_H$, the set of those twice continuously differentiable functions which vanish near the origin and near infinity. We maintain that the family of operators $H + \varepsilon V$ defined with the aid of (4.1) and (4.2) satisfies the conditions of the Concentration Theorem, i.e. conditions $(3.1)_v$, $(3.3)_v$ and $(3.4)_v$. First we claim that condition $(3.1)_v$ is satisfied. For it is easily seen that the operator V defined by (4.2) is essentially self-adjoint in $\mathfrak{D}_H$. The statement that the operator H is essentially self-adjoint in $\mathfrak{D}_H$, is a corollary of Lemma 4 of [4]. Finally the existence of a strictly self-adjoint extension of $H + \varepsilon V$ is guaranteed by a Theorem of von Neumann, [6, p. 329], which states that every formally self-adjoint real transformation over an L^2-space has a strictly self-adjoint extension. Hence the family of operators $H + \varepsilon V$ satisfies

condition $(3.1)_v$. On the other hand the point-eigenvalues and point-eigenfunctions of H have been explicitly computed, e.g. [7, p. 133]. These explicit formulae show that the operator H satisfies conditions $(3.3)_v$ and $(3.4)_v$. Therefore, the Concentration Theorem is applicable to the family of operators $H + \varepsilon V$ defined with the aid of (4.1) and (4.2).

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Appendix

The Theorem of Titchmarsh.

In [8] the Concentration Theorem is stated for the family of operators defined in section 4, however, the concentration sets, $\Delta^*(\varepsilon)$, are defined as follows :

$$(3.5)^* \qquad \Delta^*(\varepsilon) = \Delta \cup \Delta^* \bigcup \delta^*_{ij}(\varepsilon)$$

where Δ^* is any interval of the form $\Delta^* = [0,-\delta^*]$ and the intervals $\delta^*_{ij}(\varepsilon)$ are defined by

$$\delta^*_{ij}(\varepsilon) = [\lambda_i + \varepsilon < k_{ij}Vk_{ij} > - \varepsilon^p, \;\; \lambda_i + \varepsilon < k_{ij}Vk_{ij} > + \varepsilon^p] \quad .$$

Here p is an arbitrary positive number such that $p < 2$, and k_{ij} are the functions entering $(3.6)_v$.

Clearly neither of the sets, $\Delta(\varepsilon)$, defined in $(3.5)_v$ or, $\Delta^*(\varepsilon)$, defined in $(3.5)^*$, is contained in the other. Hence neither of the corresponding theorems implies the other.

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